ON THE PROBLEM OF n-STATIONARY CENTERS

(K ZADACHE n-NEPODVIZHNYKH TSENTROV)

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Among the various problems of celestial mechanics related to the n-body problem, a certain amount of interest attaches to the specific situation wherein a passive gravitational point mass M moves under the assumption that the relative disposition of the other active gravitational masses experiences no large changes.

If the attracting masses are regarded as points and if changes in the relative disposition of the attracting bodies are neglected, one arrives at the problem of the motion of the point M in a field produced by n-stationary attracting centers (the point M here represents the $(n+1)$-th body).

In addition to the problem of central motion ($n = 1$), soluble dynamics problems of this category include Euler's case [1] of two ($n = 2$) stationary Newtonian attracting centers.

This problem, which for a long time was of solely theoretical interest as an example of an integrable Liouville system [2], has recently been attracting attention in connection with the mechanics of artificial satellites, particularly after it was shown that the potential of an oblate spheroid can be approximated by the potential of two specifically chosen stationary Newtonian attracting centers [3 and 4].

The solution of the problem for n-attracting centers for $n \geq 3$ is unknown, except for a single special case of three centers pointed out by Lagrange and considered in greater detail by J.A. Serre [5].

We shall concern ourselves here with problems on the existence of periodic trajectories in the case of n-attracting centers. An arbitrary and not necessarily Newtonian gravitational law will be assumed.

Our analysis is based on the theory of quasiindices of singular force field points as set forth in [6].

1. Let the point $M(x, y)$ of unit mass ($m = 1$) situated in a field of n-attracting centers of masses m_j move along some closed orbit (C) with some constant energy h.

Assuming that the attraction forces are inversely proportional to the $(k + 1)$-th power of the distance, while the gravitational constant is equal to unity (through an appropriate choice of the units of measurement), we write out the potential $V(x, y)$ of the field under consideration,

$$V(x, y) = -\sum_{j=1}^{n} \frac{m_j}{r_j^k} \qquad \left(r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}\right) \tag{1.1}$$

Introducing into our discussion the function $\Phi(x, y) = \ln \sqrt{2(h - V(x, y))}$, which, by virtue of the energy integral, is equal to the natural logarithm of the velocity v of the point, and making use of the notion of the quasi-index J_j of the singular point O_j of the potential $V(x, y)$, we write the basic relation for periodic trajectories [6]

$$1 - J = -\frac{1}{2\pi}\iint_{(\sigma)} \Delta\Phi dxdy \qquad (J = J_1 + \ldots + J_s) \tag{1.2}$$

Here J is the sum of quasiindices s $(1 \leq s \leq n)$ of singular points O_j $(j = 1, 2, \ldots, s)$ lying within the orbit (C), (σ) is the area bounded by the orbit (C), and Δ is a Laplacian.

Making use of the notions of the density $\delta(x, y)$ and weight P of function $\Phi(x, y)$,

$$P = \iint_{(\sigma)} \delta(x, y)\, dxdy \qquad (\delta = \frac{1}{2\pi}\Delta\Phi) \tag{1.3}$$

and noting that the quasiindex J_j for the singular point $O_j(x_j, y_j)$ of the potential $V(x, y)$ (1.1) is $J_j = k/2$, by virtue of (1.2) and (1.3) we obtain

$$P = \frac{ks}{2} - 1 \qquad (1 \leqslant s \leqslant n) \tag{1.4}$$

2. The sufficient condition for the absence of periodic trajectories can be formulated either in differential form, in terms of the densities $\delta(x, y)$, or in integral form in terms of the weights P of the function $\Phi(x, y)$. As regards the necessary conditions for the existence of periodic trajectories, these will be formulated only in integral form in the weight terms of the function $\Phi(x, y)$.

The following theorems are valid.

Theorem 1. Let the region (σ) under consideration contain s $(1 \leq s \leq n)$ of the total number n of attracting centers, and let the force of attraction exerted by each center be inversely proportional to the $(k+1)$-th power of the distance. Further, let one of the conditions

$$\text{a) } ks > 2, \qquad \text{b) } ks = 2, \qquad \text{c) } ks < 2 \tag{2.1}$$

be fulfilled in the region (σ).

The sufficient condition for the absence of periodic trajectories in the region (σ) is then the stipulation that the sign of the density $\delta(x, y)$ of the function $\Phi(x, y)$ remain constant in accordance with (2.1), i.e. that

$$\text{a) } \delta(x, y) \leqslant 0, \qquad \text{b) } \delta(x, y) > 0 \ (\delta < 0), \qquad \text{c) } \delta(x, y) \geqslant 0 \tag{2.2}$$

Theorem 2. Let one of the conditions (2.1) of Theorem 1 be fulfilled. The sufficient condition for the absence of periodic trajectories in the region (σ) is then one of the following set of rules as regards the sign of the weight P of the function $\Phi(x, y)$:

$$\text{a) } P \leqslant 0, \qquad \text{b) } P > 0 \ (P < 0), \qquad \text{c) } P \geqslant 0 \tag{2.3}$$

Theorem 3. Let one of the conditions (2.1) of Theorem 1 be fulfilled. The necessary condition for the existence of periodic trajectories in the region (σ) is then one of the following set of rules as regards the sign of the weight P of the function $\Phi(x, y)$:

$$\text{a) } P > 0, \qquad \text{b) } P = 0, \qquad \text{c) } P < 0 \tag{2.4}$$

By virtue of (1.3), the proof of these theorems follows directly from basic relation (1.4).

It should be noted that the sufficient conditions for the lack of periodic trajectories (2.2) are more rigid than analogous conditions (2.3), since the former require that the sign of the density $\delta(x, y)$ be constant at all points (with the exception of the singular points O_j) of the region (σ), which is generally not required for the fulfilment of conditions (2.3).

In the case of Newtonian attracting centers, it is to be understood that $k = 1$ in all the above formulas.

3. Direct computation of the density $\delta(x, y) = \frac{1}{2}\pi\Delta\varphi(x, y)$ of the function $\Phi(x, y) = \ln\sqrt{2(h - V(x, y))}$ yields

$$\delta(x, y) = -\frac{(h - V)\Delta V + (V_x^2 + V_y^2)}{4\pi(h - V)^2} \tag{3.1}$$

where the potential $V(x, y)$ is determined in accordance with (1.1). Omitting the intervening computations, we write out the values of the quantities involved

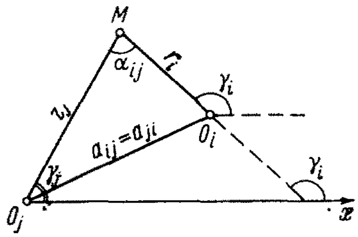

Fig. 1

$$\Delta V = -k^2 \sum_{j=1}^{n} \frac{m_j}{r_j^{k+2}} \tag{3.2}$$

$$V_x^2 + V_y^2 - V\Delta V = -\frac{k^2}{2} \sum_{i,j=1}^{n}{}^{*} \frac{m_i m_j a_{ij}^2}{r_i^{k+2} r_j^{k+2}}$$

where

$$a_{ij} = (r_i^2 + r_j^2 - 2r_i r_j \cos\alpha_{ij})^{1/2}$$

$$(a_{ij} = a_{ji},\ a_{jj} = 0) \tag{3.3}$$

is the distance between the two attracting centers O_i and O_j (Fig.1).

The final expression of the density $\delta(x, y)$ is of the form

$$\delta(x, y) = \frac{k^2}{4\pi(h - V)^2}(h\Lambda(x, y) + Y(x, y)) \tag{3.4}$$

where

$$\Lambda(x, y) = \sum_{j=1}^{n} \frac{m_j}{r_j^{k+2}}, \qquad Y(x, y) = \frac{1}{2} \sum_{i,j=1}^{n}{}^{*} \frac{m_i m_j a_{ij}^2}{r_i^{k+2} r_j^{k+2}} \tag{3.5}$$

and where an asterisk means that summation is carried out for $i \neq j$. Hence, the sign of the density δ depends intimately on the energy constant h, since

$$\operatorname{sign}(\delta) = \operatorname{sign}(h\Lambda + Y) \qquad (\Lambda, Y > 0) \tag{3.6}$$

Setting all m_i except one equal to zero, we obtain the case of one attracting center ($n = 1$). Let $m_1 = m \neq 0$. Then

$$V = -\frac{m}{r^k}, \qquad \Lambda = \frac{m}{r^{k+2}}, \qquad Y = 0$$

and the density

$$\delta = \frac{mk^2 h r^{k-2}}{4\pi(m + hr^k)^2} \tag{3.7}$$

Here $\operatorname{sign}(\delta) = \operatorname{sign}(h)$, which coincides with an earlier result [6].

Making use of the terminology adopted in the case of the two-body problem, we distinguish between three types of motion, the criterion being the value of the energy constant h: hyperbolic motion ($h > 0$), parabolic motion ($h = 0$), and elliptical motion ($h < 0$). Thus, by (3.6), the density δ for the hyperbolic and parabolic types of motion has a positive sign. By Theorem 1, this is in turn the sufficient condition for the absence of periodic trajectories for $ks \leq 2$.

In the case of elliptical motion ($h < 0$), the density $\delta(x, y)$ varies in sign. The sign of $\delta(x, y)$ changes on passage through the line described by Equation $h\Lambda(x, y) + Y(x, y) = 0$.

4. Equation $V(x, y) = h$ of the Hill curves which bound the region of possible motions of the point $M(x, y)$ by virtue of (1.1) is of the form

$$\sum_{j=1}^{n} \frac{m_j}{r_j^k} = -h \qquad (r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}) \tag{4.1}$$

The Hill curves enable us to draw some qualitative conclusions about the character of the motion. Thus, the density $\delta(x, y)$ increases without limit in absolute value as it approaches the Hill curves (4.1). By virtue of (3.4) and (4.1), it assumes infinitely large values on the curves themselves.

Let us consider the case of elliptical motion ($h < 0$) for large absolute values of the energy constant h. The Hill curve equation (4.1) implies here that one of the values r_j, e.g. r_i, must be very small, while the rest of the r_j $(j \neq i)$ have finite values since the inequalities $r_i + r_j \geq a_{ij}$ $(j \neq i)$ are valid throughout the entire period of motion. Hence, for large $|h|$ Hill's curve consists of n (according to the number of attracting centers) oval-shaped curves of very small linear dimensions, each of which includes an attracting center. Elliptical motion in the case of n attracting bodies and large $|h|$ involves the phenomenon of "capture", and the point M moves within one of these ovals (the one in which it lay at the initial instant).

The oval in which capture occurs can be approximately represented as a circle of small radius r_i given by Expression

$$r_i \sim \left(\frac{m_i}{-h - b_i}\right)^{1/k} \qquad \left(-h > 0,\ b_i = \sum_{j=1}^{n}{}^{*} \frac{m_j}{a_{ij}{}^k}\right) \tag{4.2}$$

where m_i is the mass of the attracting center O_i within the oval. The asterisk indicates that $j \neq i$ in the summation.

5. Let us consider the problem of the existence of periodic trajectories in the neighborhood of some attracting center, let us say O_1, in the presence of other attracting centers O_j $(j = 2, 3, \ldots, n)$.

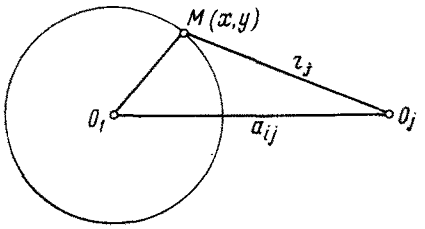

Fig. 2

Placing the origin of our coordinate system at the point O_1 (Fig.2) and denoting r_1 by r and m_1 by m, let us write out the values of the quantities appearing in Expression (3.4) for the density $\delta(x, y)$. We have

$$V(x, y) = -\frac{m}{r^k}(1 + r^k F_1(x, y))$$

$$\Lambda(x, y) = \frac{m}{r^{k+2}}(1 + r^{k+2} F_2(x, y))$$

$$Y(x, y) = \frac{m}{r^{k+2}} \sum_{j=2}^{n} \frac{m_j a_{1j}{}^2}{r_j{}^{k+2}} + F_3(x, y)$$

where $F_1(x, y)$, $F_2(x, y)$ and $F_3(x, y)$ are entire functions of x and y. Limiting ourselves to the leading terms in the indicated expansions, we obtain the following approximate value for the density:

$$\delta = \frac{mk^2 (h + b_1) r^{k-2}}{4\pi (m + hr^k)^2} \qquad \left(b_1 = \sum_{j=2}^{n} \frac{m_j}{a_{1j}{}^k}\right) \tag{5.1}$$

so that

$$\operatorname{sign}(\delta) = \operatorname{sign}(h + b_1)$$

Hence, by virtue of Theorem 1 and depending on the values of the exponent k, to wit: (1) $k > 2$, (2) $k = 2$, and (3) $k < 2$, the sufficient conditions for the absence of periodic trajectories in the neighborhood of the attracting center O_1 is the fulfilment of one of the following conditions, respectively:

$$1)\ h + b_1 \leqslant 0, \qquad 2)\ h + b_1 \neq 0, \qquad 3)\ h + b_1 \geqslant 0$$

If the relations

$$1)\ h + b_1 > 0, \qquad 2)\ h + b_1 = 0, \qquad 3)\ h + b_1 < 0$$

are valid with the values of h indicated above, then, by virtue of Theorem 3, the necessary (but unsufficient) conditions for the existence of periodic trajectories are fulfilled.

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